

A NOTE ON COMPACT MARKOV OPERATORS

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ABSTRACT. The analytic properties of the Markov operator associated to a random walk are common tools in the study of the behaviour and some probabilistic features related to the walk. In this paper we consider a class of Markov operators which generalizes the class of compact Markov operators and we study some probabilistic properties of the associated random walk.

1. BASIC DEFINITIONS

Let (X, P) be an irreducible, random walk on the state space X which is at most countable. We suppose that the (usually infinite) stochastic matrix P describes a Markov chain $\{Z_n\}_{n \in \mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with transition probabilities $p(x, y) := \mathbb{P}[Z_{n+1} = y | Z_n = x]$ homogeneous in time. Besides we consider the n -step transition probabilities $\{p^{(n)}(x, y)\}_{x, y \in X}$ which represent the stochastic matrix associated to the n -th convolution power of P .

The *Markov operator* associated to the random walk is defined as follows

$$(1) \quad \begin{aligned} D(P) &:= \left\{ f : X \rightarrow \mathbb{R} : \sum_{y \in X} p(x, y) |f(y)| < +\infty, \forall x \in X \right\}, \\ (Pf)(x) &:= \sum_{y \in X} p(x, y) f(y), \quad \forall f \in D(P), \forall x \in X; \end{aligned}$$

note that $\mathcal{D}(P) \supseteq l^\infty(X)$ and that $P|_{l^\infty(X)}$ is a bounded linear operator from $l^\infty(X)$ into itself.

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To explore the behaviour of the random walk (X, P) and its main properties we introduce the two generating functions

$$G(x, y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)z^n, \quad F(x, y|z) = \sum_{n=0}^{\infty} f^{(n)}(x, y)z^n$$

where $\{f^{(n)}(x, y)\}_{x, y \in X}$ are the first time return probabilities, namely $f^{(n)}(x, y) = \mathbb{P}(Z_n = y, Z_i \neq y, \forall i = 1, \dots, n-1 | Z_0 = x)$, $f^{(0)}(x, y) = 0$. Both the generating functions must be considered inside their circle of convergence in \mathbb{C} .

An irreducible random walk (X, P) is called *transient* if and only if there exists (\Leftrightarrow for any) $x \in X$ such that $F(x, x) < 1$ and *recurrent* otherwise. Among the recurrent random walks we distinguish the class of *positive recurrent* and *null recurrent* depending on whether $\bar{\tau}_x := \sum_{n=1}^{\infty} n f^{(n)}(x, x) < +\infty$ for some (\Leftrightarrow for any) $x \in X$ or not.

We note that positive recurrence is a strong assumption: for instance if (X, P) is the simple random walk on a infinite, locally finite, non-oriented, connected graph (X, E) , then it is not positive recurrent. Indeed it is easily reversible with reversibility measure given by $m(x) := \#\{y : (x, y) \in E\}$, which is clearly infinite. According to Theorem 1.18 of [1], if an irreducible Markov chain is recurrent, then it admits a unique (up to multiplication) stationary measure and this one is finite if and only if the walk is positive recurrent. Since a reversibility measure is stationary, if the walk were positive recurrent, then m should be finite.

The importance of this class of random walks is highlighted by Theorem 1.18 of [1] (see also Theorem 3.2 of [2]).

Remark 1.1. *We note that (X, P) is positive recurrent if and only if there exists (\Leftrightarrow for all) $x \in X$,*

$$\lim_{z \rightarrow 1^-} \frac{1 - F(x, x|z)}{1 - z} < +\infty.$$

Just take in mind that the limit always exists (finite or infinite) due to the decomposition

$$\lim_{z \rightarrow 1^-} \frac{1 - F(x, x|z)}{1 - z} = \lim_{z \rightarrow 1^-} \frac{1 - F(x, x|1)}{1 - z} + \lim_{z \rightarrow 1^-} \frac{F(x, x|1) - F(x, x|z)}{1 - z},$$

and, using well-known arguments, we have

$$\begin{aligned} \sum_{n=0}^{\infty} n f^{(n)}(x, x) &= \lim_{z \rightarrow 1^-} \sum_{n=0}^{\infty} n f^{(n)}(x, x) z^n \\ &= \lim_{z \rightarrow 1^-} F'(x, x|z) = \lim_{z \rightarrow 1^-} \frac{F(x, x|1) - F(x, x|z)}{1 - z}; \end{aligned}$$

(note that the last equality holds also if the limit is $+\infty$).

2. COMPACT MARKOV OPERATORS

In this section we want to study the behaviour of a random walk whose associated Markov operator satisfying equation (2) below. In particular we study compact Markov operators.

We recall here the characterization of a compact operator (with non-negative matrix elements) defined by equation (1) (see [2], Theorem 2.2).

Theorem 2.1. *Let X be a countable set and let P be a transition operator on X with non negative elements, satisfying the condition $\sup_{x \in X} \sum_{y \in X} p(x, y) < +\infty$. Then P is a bounded, linear operator from $l^\infty(X)$ into itself; moreover $P : l^\infty \mapsto l^\infty$ is compact if and only if for any given $\epsilon > 0$ there exists a finite subset $A_\epsilon \subset X$ such that $\sup_{x \in X} \sum_{y \in X \setminus A_\epsilon} p(x, y) < \epsilon$.*

The next Theorem is the main result of this section: its corollary enhances Proposition 2.3 of [2].

Theorem 2.2. *Let (X, P) be an irreducible Markov chain, and suppose that there exists $\epsilon \in (0, 1)$ and a finite subset $A \subset X$ such that*

$$(2) \quad \sup_{x \in X} \sum_{y \in X \setminus A} p(x, y) < \epsilon;$$

then (X, P) is positive recurrent.

Before looking at the proof, we want to understand what equation (2) implies from the point of view of the walker.

Let us consider the preadjoint map $P_* : l^1(X) \rightarrow l^1(X)$ acting as

$$P_*\nu(y) \equiv \nu P(y) := \sum_{x \in X} \nu(x)p(x, y), \quad \forall y \in X.$$

This is a mass-preserving map, indeed $\sum_{y \in X} \nu P(y) = \sum_{x \in X} \nu(x)$; moreover $\nu \geq 0$ implies $P_*\nu \geq 0$. If ν is the probability distribution of the position of the walker at a certain time, then $P_*\nu$ is the probability distribution after one step.

In term of this evolution map, equation (2) is equivalent to the existence of a finite subset A such that, given any probability distribution ν , the probability distribution after one step satisfies $P_*\nu(A) \geq 1 - \epsilon$ (or, equivalently, $P_*^n \nu(A) \geq 1 - \epsilon$ for any $n \in \mathbb{N}^*$).

Since from the Law of large numbers, for any given $x \in X$, \mathbb{P} -a.c.

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x\}}(Z_n) = \begin{cases} 0 & \text{in the transiente or null-recurrent case} \\ \frac{1}{\bar{\tau}_x} & \text{in the positive recurrent case} \end{cases}$$

and since in the positive recurrent case, $1/\bar{\tau}_x$ represents the unique stationary probability measure (see [3], Section I.7, Theorem 1), hence the walker will pass (asymptotically) at least $1 - \epsilon$ of its time in A .

Proof. (of Theorem 2.2). Let A and ϵ satisfying equation (2).

Let us note that for any given $x \in X$ we have

$$\sup_{x \in X} \sum_{y \in X \setminus A} p^{(n)}(x, y) < \epsilon,$$

indeed for any given $n \in \mathbb{N}^*$

$$\mathbb{P}(Z_n \in A | Z_0 = x) = \sum_{z \in X} \mathbb{P}(Z_n \in A | Z_{n-1} = z) \mathbb{P}(Z_{n-1} = z | Z_0 = x) \geq 1 - \epsilon.$$

Let $x_0 \in X \setminus A$ be fixed and rewrite the previous equation as

$$\sum_{x \in A} p^{(n)}(x_0, x) \geq 1 - \epsilon, \quad \forall n \in \mathbb{N}^*.$$

This implies

$$\sum_{x \in A} G(x_0, x | z) \geq (1 - \epsilon) \left(\sum_{j=1}^{\infty} z^j \right) = \frac{z(1 - \epsilon)}{1 - z}.$$

Since $G(x, y | z) = \delta_{xy} + F(x, y | z)G(y, y | z) = \delta_{xy} + F(x, y | z)/(1 - F(y, y | z))$, the previous inequality becomes

$$\sum_{x \in A} \frac{F(x_0, x | z)}{1 - F(x, x | z)} \geq \frac{z(1 - \epsilon)}{1 - z}.$$

Now, taking in mind the usual position $1/\infty = 0$, since A is finite,

$$0 < 1 - \epsilon \leq \lim_{z \rightarrow 1^-} \sum_{x \in A} \frac{(1 - z)F(x_0, x | z)}{1 - F(x, x | z)} = \sum_{x \in A} \frac{F(x_0, x | 1)}{\lim_{z \rightarrow 1^-} (1 - F(x, x | z))/(1 - z)}.$$

This easily implies the existence of $x_1 \in A$ such that

$$\lim_{z \rightarrow 1^-} \frac{1 - F(x_1, x_1 | z)}{1 - z} < +\infty$$

hence, by Remark 1.1, (X, P) is positive recurrent. \square

Corollary 2.3. *Let (X, P) be an irreducible Markov chain, such that P is compact; then (X, P) is positive recurrent.*

We emphasize that the Markov operator associated to a positive recurrent random walk needs not to be compact as the following example shows. Take $X = \mathbb{N}$ and define the transition probabilities as follows:

$$\begin{cases} p(0, 1) = 1, \\ p(n, n+1) = 1-p, & \forall n \in \mathbb{N}^*, \\ p(n+1, n) = p, & \forall n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where $p \in (0, 1)$. The first time return probabilities generator function F can be easily calculated for $x = y = 0$ as

$$F(0, 0|z) = \frac{2pz^2}{1 + \sqrt{1 - 4z^2p(1-p)}};$$

the corresponding random walk is transient if $p \in (0, 1/2)$, null recurrent if $p = 1/2$ and positive recurrent if $p \in (1/2, 1)$, but equation (2) does not hold, hence the Markov operator is always non compact.

3. SOME ESTIMATES

Let us define the time of the first return onto the vertex $x \in X$ as $T_x := \inf\{n \geq 1 : Z_n = x\}$; in the recurrent case we have

$$\bar{\tau}_x = \mathbb{E}[T_x | Z_0 = x] = \lim_{z \rightarrow 1^-} \frac{1 - F(x, x|z)}{1 - z} \equiv \lim_{z \rightarrow 1^-} (1 - z)G(x, x|z).$$

Moreover if equation (2) holds, we have that $F(x, y) = 1$ for any $x, y \in X$ and

$$1 \geq \sum_{x \in A} \frac{1}{\bar{\tau}_x} \geq 1 - \epsilon$$

which implies

$$\min_{x \in A} \bar{\tau}_x \leq \frac{\text{card}(A)}{1 - \epsilon}.$$

Besides for the first time entrance in A , $T_A := \inf\{n > 0 : Z_n \in A\}$, the following hold

$$\mathbb{P}(T_A \geq n | Z_0 = x) \leq \epsilon^{n-1}, \quad \mathbb{E}[T_A | Z_0 = x] \leq \frac{1}{1 - \epsilon}.$$

In the reversible case, it is possible to find lower bounds for the n -step transition probabilities $p^{(n)}(x, x)$ and their generating function. In this case the reversibility measure m satisfies $m(x) \propto 1/\bar{\tau}_x$.

Lemma 3.1. *Let (X, P) be a reversible random walk, m a reversibility measure and $x \in \Gamma$ a fixed vertex. If there exists $n \in \mathbb{N}^*$ and $A \subseteq X$ such that*

$$\sum_{y \in X \setminus A} p^{(n)}(x, y) \leq \epsilon,$$

then

$$p^{(2n)}(x, x) \geq (1 - \epsilon)^2 \frac{m(x)}{m(A)}.$$

The easy proof of this lemma is straightforward and we omit it. By using this result one see immediately that, for any $x \in A$

$$G(x, x|z) \geq \frac{(1 - \epsilon)^2}{1 - z^2} \frac{m(x)}{m(A)}, \quad z \in [0, 1).$$

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